A methodology for analyzing progressive damage accumulation on multiple spatial scales (micro- and macro-scale) in composite materials is presented in this paper. Idealization (homogenization) of heterogeneous media and evolution of damage on micro- and macro-scales are considered simultaneously at each incremental analysis step. The classical mathematical homogenization theory is extended to account for damage effects on distinct spatial scales through the introduction of an asymptotic expansion of damage parameter (or damage tensor in general). Local solutions on micro-scale provide the homogenized material properties that a global structure behaves on the macro-scales. The responses in the local fields, i.e. microscopic phases, can be reconstructed through the scale linking relations along with the global responses as input. The application of this multi-scale simulation method to composite patch repair for offshore structures is demonstrated by numerical examples.

INTRODUCTION

Use of composite materials for various marine and offshore structures attracted active research and development activities in the past two decades. The covered areas have been extended way beyond the traditional light-duty structures and into such important applications as composite mooring lines, tendons and tethers, composite risers, topside piping systems, air accumulators of riser tensioning systems, and composite patch repairs, to name a few (Lo, et al., 2001, Salama, et al, 2000; Turton, et al, 2005). To reduce the amount of expensive qualification tests as required by most of current practices, it is imperative to develop efficient and physically robust virtual prototyping tools to verify the design and identify any potential failure mechanisms/modes.

Composite structures used in naval and offshore applications are typically in the form of laminated fiber reinforced plastics. Although finite element analysis using laminated plate, shell or solid elements, which are available in most commercial FEA software, has been widely adopted to predict the response and failure of composite structures, the accuracy of the analysis is greatly affected by the lamina constitutive model that users must supply. This paper presents an efficient material modeling approach with special attention on the damage accumulation induced by thermo-mechanical loading.

Damage in composite materials occurs through different mechanisms that are complex and usually involve strong interactions between micro-constituents (Dvorak, 2000; Krajcinovic, 1996; Talreja, 1989). During the past two decades, a number of models have been developed to simulate damage and failure process in composite materials, among which the damage mechanics approach attracted significant attention because it provides a viable framework for the description of the distributed damage including material stiffness degradation, initiation, growth and coalescence of microcracks and voids with irregular shape, size and orientations.

The multi-scale damage modeling approach presented in this paper is to simultaneously carry out idealization (homogenization) of heterogeneous media and evolution of damage on micro- and macro-scales. The classical mathematical homogenization theory
(Bakhvalov and Panassenko, 1989; Guedes and Kikuchi, 1990) is extended to account for damage effects on distinct spatial scales through the introduction of an asymptotic expansion of damage parameter (or damage tensor in general), which leads to the derivation of the closed form solutions relating local fields in microscopic phases to overall strains and damage of global structure. With these scale-linking relations, the finite element analysis of progressive damage can be conducted on the level of global structures with coarse mesh. The material properties of the global structure are updated consistently along with the damage accumulation in microscopic phases.

A nonlocal theory is developed using the concept of nonlocal phase fields (stress, strain, free energy density, damage release rate, etc.). Those nonlocal phase fields are defined as weighted averages over each phase in the characteristic volume in a manner analogous to that currently practiced in concrete (Bazant and Pijaudier-Cabot, 1988), with the only exception being that the piecewise weight function is adopted. On the global (macro) level, the finite element size has to be limited in order to ensure a valid use of the mathematical homogenization theory and to limit localization.

Numerical examples for composite patch repair are conducted to demonstrate the capability of the proposed approach.

**MATHEMATICAL HOMOGENIZATION FOR DAMAGED COMPOSITES SUBJECTED TO THERMOECHANICAL LOAD**

In this section the classical mathematical homogenization theory for statistically homogeneous composite media is extended to account for damage effects. The strain-based continuum damage theory is adopted for constructing the constitutive relations at the level of micro-constituents. Closed form expressions of local strain and stress fields in a multi-phase composite media are derived. Attention is restricted to small deformations.

The microstructure of a composite material is assumed to be locally periodic (Y-periodic) within a Representative Volume Element (RVE), denoted by \( \Theta\), which is statistically homogeneous. Let \( \bar{x}\) be a macroscopic coordinate vector in macro domain \( \Omega\) and \( \bar{y}=\bar{x}/\bar{\zeta}\) be a microscopic position vector in \( \Theta\). Here, \( \bar{\zeta}\) denotes a very small positive number compared with the dimension of \( \Omega\) and \( \bar{y}=\bar{x}/\bar{\zeta}\) is regarded as a stretched coordinate vector in the microscopic domain. When a solid is subjected to load and boundary conditions, the resulting deformation, stresses, and internal variables may vary from point to point within the SHV due to the high level of heterogeneity. All quantities are assumed to have two explicit dependences: one on the macroscopic level \( \bar{x}\) and the other one, on the level of micro-constituents \( y=\bar{x}/\bar{\zeta}\).

For any Y-periodic response function \( f \), there exists \( f(\bar{x},\bar{y}) = f(\bar{x},\bar{y} + k\bar{\zeta}) \) in which vector \( \hat{k}\) is the basic period of the microstructure and \( k\) is a 3 by 3 diagonal matrix with integer components. Adopting the classical nomenclature, any Y-periodic function \( f \) can be represented as \( f(\bar{x}) = f(\bar{x},\bar{y}(\bar{x})), \) where superscript \( \bar{\zeta}\) denotes a Y-periodic function \( f \). The indirect macroscopic spatial derivatives of \( f_\zeta \) can be calculated by the chain rule as

\[
\tilde{f}_\zeta(\bar{x}) = f_\zeta(x,y) = \frac{1}{\zeta} \tilde{f}_\zeta(x,y)
\]

where the comma followed by a subscript variable \( z \), denotes a partial derivative with respect to the subscript variable (i.e. \( f_\zeta = \partial f/\partial x_i \)). Summation convention for repeated subscripts is employed, except for subscripts \( x \) and \( y \).

The constitutive equation on the microscale is derived from continuum damage theory based on the thermodynamics of irreversible processes and internal state variable theory. To model the isotropic damage process on micro-scale, a scalar damage parameter \( \omega^\zeta \) is defined as a function of microscopic and macroscopic position vectors, i.e., \( \omega^\zeta = \omega(x,y) \).

It is assumed that the micro-constituents possess homogeneous properties and satisfy equilibrium, constitutive, kinematics and compatibility equations as well as jump conditions at the interface between the micro-phases. The corresponding boundary value problem is governed by the following set of equations:

\[
\begin{align*}
\sigma^\zeta_{ij,\bar{x}_j} + b_i &= 0 \quad \text{in} \quad \Omega \quad (2) \\
\sigma^\zeta_{ij} &= (1 - \omega^\zeta)L_{ijkl}(\varepsilon^\zeta_{ij} - E_{ijkl}T) \quad \text{in} \quad \Omega \quad (3) \\
\varepsilon^\zeta_{ij} &= u^\zeta_{\bar{x}_i,\bar{x}_j} \quad \text{in} \quad \Omega \quad (4) \\
u^\zeta &= n_i \quad \text{on} \quad \Gamma_u \quad (5) \\
\sigma^\zeta_{ij} n_j &= -t_i \quad \text{on} \quad \Gamma_t \quad (6)
\end{align*}
\]

where \( \omega^\zeta \in [0,1) \) is a scalar damage parameter governed by a strain history parameter; \( \varepsilon^\zeta_{ij} \) and \( \sigma^\zeta_{ij} \) are components of stress and strain tensors; \( L_{ijkl} \) represents components of elastic stiffness satisfying symmetry and positivity conditions; \( E_{ijkl} \) is anisotropic coefficient of thermal expansion and \( T \) denotes the steady state temperature field; \( b_i \) is a body force assumed to be independent of \( \bar{x}\); \( u^\zeta_{\bar{x}_i} \) denotes the components of the displacement vector; the subscript pairs with parentheses denote the symmetric gradients defined as \( u^\zeta_{\bar{x}_i,\bar{x}_j} = \frac{1}{2}(u^\zeta_{\bar{x}_i,\bar{x}_j} + u^\zeta_{\bar{x}_j,\bar{x}_i}) \); \( \Omega \) denotes the macroscopic domain of interest with boundary \( \Gamma \); \( \Gamma_u \) and \( \Gamma_t \) are
boundary portions where displacements \( \hat{u}_i \) and tractions \( \hat{t}_i \) are prescribed, respectively, such that \( \Gamma_u \cap \Gamma_i = \emptyset \) and \( \Gamma = \Gamma_u \cup \Gamma_i \); \( \hat{\nu}_i \) denotes the normal vector on \( \Gamma \). The interface between the phases is assumed to be perfectly bonded, i.e. \( [\sigma_i]\hat{\nu}_i = 0 \) and \( [u] \hat{\nu}_i = 0 \) at the interface \( \Gamma^{\text{int}} \), where \( \hat{\nu}_i \) is the normal vector to \( \Gamma^{\text{int}} \) and \( \hat{n}_i \) is a jump operator.

It is apparently not feasible to use a brute force numerical approach attempting at discretization of the entire macroscopic domain with a grid spacing smaller than the characteristic size of microscale features (such as fibers). Instead, a mathematical homogenization method based on the double-scale asymptotic expansion is employed in this paper to account for microstructural effects on the macroscopic response without explicitly representing the details of microstructure in the global analysis. The double-scale asymptotic expansions on \( \Omega \times \Theta \) are applied to the displacement field, \( u^\ell(x) = u^\ell(x,y) \), and the damage parameter, \( \omega^\ell(x) = \omega(x,y) \):

\[
\begin{align*}
    u^\ell(x,y) &= u_0^\ell(x,y) + \zeta u_1^\ell(x,y) + \ldots \\
    \omega(x,y) &= \omega_0(x,y) + \zeta \omega_1(x,y) + \ldots
\end{align*}
\]

(7)

(8)

The corresponding strain expansions on \( \Omega \times \Theta \) can be obtained by substituting (7) into (4) with consideration of the indirect differentiation rule (1)

\[
\begin{align*}
    \varepsilon^\ell_{ij}(x,y) &= \frac{1}{\zeta} \varepsilon_{ij}^{-1}(x,y) + \varepsilon_0^{\ell}_{ij}(x,y) + \zeta \varepsilon_1^{\ell}_{ij}(x,y) + \ldots
\end{align*}
\]

(9)

where strain components for the two lowest orders of \( \zeta \) are given as

\[
\begin{align*}
    \varepsilon_{ij}^{-1} &= \varepsilon_{ij}(\mathbf{u}^0) , \quad \varepsilon_0^{\ell}_{ij} = \varepsilon_{ij}(\mathbf{u}^0) + \varepsilon_{ij}(\mathbf{u}^1) \\
    \varepsilon_{x0}^{\ell}(\mathbf{u}^0) &= u_0^{\ell}_{(x,0)} ; \quad \varepsilon_{y0}^{\ell}(\mathbf{u}^0) = u_0^{\ell}_{(y,0)} ; \quad \varepsilon_{y0}^{\ell}(\mathbf{u}^1) = u_1^{\ell}_{(y,0)}
\end{align*}
\]

(10)

Stresses and strains for different orders of \( \zeta \) are related by the constitutive equation (3)

\[
\begin{align*}
    \sigma^\ell_{ij} &= (1 - \omega^0)L_{ijkl}\varepsilon^\ell_{kl} \\
    \sigma_0^{\ell}_{ij} &= (1 - \omega^0)L_{ijkl}(\varepsilon^{\ell}_{kl} - E_{kl}T) - \omega^1L_{ijkl}\varepsilon^{-1}_{kl} \\
    \sigma_1^{\ell}_{ij} &= (1 - \omega^0)L_{ijkl}\varepsilon^{-1}_{kl} - \omega^1L_{ijkl}(\varepsilon^{\ell}_{kl} - E_{kl}T) - \omega^2L_{ijkl}\varepsilon^{-1}_{kl}
\end{align*}
\]

(11)

The resulting asymptotic expansion of stress is given as

\[
\begin{align*}
    \sigma(x,y) &= \frac{1}{\zeta} \sigma_{ij}^{-1}(x,y) + \sigma_0^{\ell}_{ij}(x,y) + \zeta \sigma_1^{\ell}_{ij}(x,y) + \ldots
\end{align*}
\]

(12)

Inserting the stress expansion (12) into equilibrium equation (2) and making use of equation (1) yield the following equilibrium equations for various orders up to \( O(\zeta^2) \):

\[
\begin{align*}
    O(\zeta^{-2}): & \quad \sigma_{ij}^{-1} = 0 \\
    O(\zeta^{-1}): & \quad \sigma_0^{\ell}_{ij} + \sigma_1^{\ell}_{ij} = 0 \\
    O(\zeta^0): & \quad \sigma_0^{\ell}_{ij} + \sigma_1^{\ell}_{ij} + b_1 = 0
\end{align*}
\]

(13)

From the \( O(\zeta^2) \) equilibrium equation, the classic solution

\[
\varepsilon_{xy}(\mathbf{u}^0) = u_0^{(y,xy)} = 0 \quad u_0^{(x)} = u_0^0(x)
\]

(14)

can be obtained, and therefore

\[
\sigma_{ij}^{-1}(x,y) = \varepsilon_{ij}(x,y) = 0
\]

(15)

For \( O(\zeta^{-1}) \) equilibrium equation in (13), it can be re-formulated using the relations in (10) and (11):

\[
\{ (1 - \omega^0)L_{ijkl}(\varepsilon_0^{\ell}(\mathbf{u}^0) + \varepsilon_1^{\ell}(\mathbf{u}^1) - E_{kl}T) \}_{,ij} = 0 \quad \text{in} \quad \Theta
\]

(16)

To solve (16) up to a constant, the following separation of variables is introduced

\[
\varepsilon_0^{\ell}(\mathbf{u}^0) = H_{ijkl}(y)\{ \varepsilon_{x0}^{\ell}(\mathbf{u}^0) + d_{ij}^{\ell}(x) \}
\]

(17)

where \( H_{ijkl} \) is a Y-periodic function determined by the geometric and material properties in \( \Theta \); \( d_{ij}^{\ell}(x) \) is the macroscopic damage-induced strain (eigenstrain) driven by the macroscopic strain \( \varepsilon_{kl}^{\ell} \), which can be defined as follows after taking in account of (14).

\[
\varepsilon_{kl}^{\ell} = \varepsilon_{kl}^{\ell}(\mathbf{u}^0)
\]

(18)

It can be stated that if \( \varepsilon_{kl}^{\ell} = 0 \), then \( d_{ij}^{\ell}(x) = 0 \) and \( \omega_0(x,y) = 0 \). Note that vice versa is not true, i.e., if \( d_{ij}^{\ell}(x) = 0 \) or \( \omega_0(x,y) = 0 \), the macroscopic strain \( \varepsilon_{kl}^{\ell} \) may not necessarily be zero. In (17) both \( H_{ijkl} \) and \( d_{ij}^{\ell}(x) \) are symmetric with respect to indices \( k \) and \( l \).

Based on the decomposition given in (17), \( O(\zeta^{-1}) \) equilibrium equation takes the following form:

\[
\{ (1 - \omega^0)L_{ijkl}[I_{klimn} + G_{klmn}(\mathbf{u}^0) - E_{kl}T + G_{klimn}d_{mn}^{\ell}(x)] \}_{,ij} = 0 \quad \text{in} \quad \Theta
\]

(19)

where

\[
I_{klimn} = \frac{1}{2}(\delta_{mk}\delta_{nl} + \delta_{mk}\delta_{ml}), \quad G_{klmn}(y) = H_{(k,y)mn}(y)
\]

(20)

and \( \delta_{mk} \) is the Kronecker delta, while \( G_{klmn} \) is known as a polarization function. It can be shown that the integrals of the polarization functions in \( \Theta \) vanish due to periodicity conditions. Since equation (19) should be valid for arbitrary macroscopic fields without dependence on \( x \), it can be determined by taking
\[ d_{k}^{\text{m}}(x) = 0 \quad \text{(and } \omega = 0 \text{ but } \hat{\varepsilon}_{kl} \neq 0\text{)}, \]
which yields the following equation defined in microscopic domain \( \Theta \):
\[ \{ L_{ijkl}(x_{kl} + H_{(k,\ell)mn}) \}_{j,\ell} = 0 \tag{21} \]
Equation (21) together with the Y-periodic boundary conditions is a classic linear boundary value problem in \( \Theta \). By exploiting the symmetry of \( (m, n) \), it can be easily solved using conventional finite element method for 3 right hand side vectors in 2-D and 6 in 3-D (Guedes and Kikuchi, 1990).

In absence of damage, the strain asymptotic expansion (9) can be expressed in terms of the macroscopic strain \( \bar{\varepsilon}_{ij} \) as
\[ \varepsilon_{ij} = A_{ijkl} \bar{\varepsilon}_{kl} + O(\zeta); \quad A_{ijkl} = I_{ijkl} + G_{ijkl} \tag{22} \]
where \( A_{ijkl} \) is typically termed as elastic strain concentration function within the context of Micromechanics.

The elastic homogenized stiffness \( L_{ijkl} \) follows from \( O(\zeta) \) equilibrium equation:
\[ L_{ijkl} = \frac{1}{|\Theta|} \int_{\Theta} L_{ijmn} A_{mnlk} d\Theta = \frac{1}{|\Theta|} \int_{\Theta} A_{mnij} L_{mnsr} A_{srlk} d\Theta \tag{23} \]
where \( |\Theta| \) is the volume of a RVE.

After solving (21) for \( H_{ijmn} \), eigenstrain \( d_{mn}^{\omega} \) can be calculated from (19) such that
\[ d_{mn}^{\omega}(x) = \left. \Theta \right| (1 - \omega^{0}) G_{ijkl} L_{ijkl} G_{klmn} d\Theta \quad \{ \left. \Theta \right| (1 - \omega^{0}) G_{ijkl} L_{ijkl} A_{klmn} d\Theta \} \varepsilon_{mn} \quad \{ \left. \Theta \right| (1 - \omega^{0}) G_{ijkl} L_{ijkl} E_{kl} d\Theta \} \} \tag{24} \]
Let \( \Psi = \{ \psi^{(\eta)}(y) \} \) be a set of \( C^{-1} \) continuous functions, then the damage parameter \( \omega^{\eta}(x, y) \) is assumed to have the following decomposition
\[ \omega^{\eta}(x, y) = \sum_{n}^{\eta} \psi^{(n)}(y) \omega^{(n)}(x) \tag{25} \]
where \( \psi^{(\eta)}(y) \) is a damage distribution function on the microscale. Rewriting (24) in terms of strain concentration function \( A_{ijkl} \) and manipulating it with (23) and (25) yield
\[ d_{kl}^{\omega}(x) = \Psi_{km}^{\omega}(x) \varepsilon_{mn} - D_{kl}^{\omega}(x) T \tag{26} \]
where
\[ D_{kl}^{\omega}(x) = I_{klst} - n_{ij}^{\omega}(x) \omega^{(n)}(x) C_{st}^{(n)}(x) \]
\[ B_{ij}^{(n)} = \frac{1}{|\Theta|} \int_{\Theta} (L_{ijmn} - L_{ijmn})^{-1} \psi^{(n)} G_{ijkl}^{(n)} L_{ijkl} G_{ijkl}^{st} d\Theta \]
\[ C_{ijkl}^{(n)} = \frac{1}{|\Theta|} \int_{\Theta} (L_{ijmn} - L_{ijmn})^{-1} \psi^{(n)} G_{ijkl}^{(n)} L_{ijkl} G_{ijkl}^{st} A_{ijkl} d\Theta \]
\[ L_{ijmn} = \frac{1}{|\Theta|} \int_{\Theta} L_{ijkl} d\Theta \tag{27} \]
and
\[ D_{ij}(x) = I_{klst} - n_{ij}^{\omega}(x) \omega^{(n)}(x) - D_{ij}^{\omega}(x) \]
\[ P_{ij} = \frac{1}{|\Theta|} \int_{\Theta} (L_{ijmn} - L_{ijmn})^{-1} G_{ijkl}^{(n)} L_{ijkl} G_{ijkl}^{st} E_{ij} d\Theta \]
\[ G_{ij}^{(n)} = \frac{1}{|\Theta|} \int_{\Theta} (L_{ijmn} - L_{ijmn})^{-1} \psi^{(n)}(y) G_{ijkl}^{(n)} L_{ijkl} G_{ijkl}^{st} E_{ij} d\Theta \tag{28} \]
In conjunction with (17) and (26), the asymptotic expansion of strain field (9) can be finally cast as
\[ \varepsilon_{ij}(x, y) = A_{ijmn}(x, y) \bar{\varepsilon}_{mn}(x) + G_{ij}(y) d_{mn}^{\omega}(x) + O(\zeta) \]
\[ = A_{ijmn}(x, y) \bar{\varepsilon}_{mn}(x) + G_{ij}(y) D_{kl}^{\omega}(x) \varepsilon_{mn}(x) \]
\[ - G_{ijkl}(y) D_{ij}(x) T + O(\zeta) \tag{29} \]
where \( G_{ij}(y) \) can be interpreted as a damage strain influence function. Note that the asymptotic expansion of the strain field is given as a sum of mechanical fields induced by the macroscopic strain via elastic strain concentration function and thermodynamical fields governed by damage-induced strain \( d_{kl}^{\omega}(x) \) through the damage strain influence function.

Finally, the equilibrium equation in macroscopic domain is obtained by integrating the \( O(\zeta) \) equilibrium equation (13) over \( \Theta \) and using the constitutive relation (11) and the asymptotic expansion of strain field (29).

The term vanishes due to periodicity so that:
\[ \sigma_{ij,xy} + b_{ij} = 0 \quad \text{and} \quad \sigma_{ij} = L_{ijmn} \bar{\varepsilon}_{mn} - F_{ij} T \tag{30} \]
where macroscopic stress
\[ \sigma_{ij} = \frac{1}{|\Theta|} \int_{\Theta} \sigma_{ij}^{\omega} d\Theta \]
\( L_{ijmn} \) is an instantaneous secant stiffness given as
\[ L_{ijmn} = L_{ijkl} \sum_{n}^{\eta} \omega^{(n)}(y) L_{ijkl} A_{klmn} d\Theta \]
\[ - L_{ijkl} \sum_{n}^{\eta} \omega^{(n)}(y) L_{ijkl} d\Theta \quad \text{and} \quad D_{klmn} \]
and $F_{ij}$ is an instantaneous thermal-stress coefficient tensor.

$$
P_{ij} = \Theta_{ijkl} F_{ij} - \frac{1}{\Theta} \sum_{\eta=1}^{m} \varphi^{(\eta)}(y) \Theta_{ijkl} L_{ijkl} F_{ij} \Theta \eta + \frac{1}{\Theta} \sum_{\eta=1}^{m} \varphi^{(\eta)}(y) L_{ijkl} \Theta_{ijkl} \Theta \eta (L_{ijkl} \Theta \eta)
$$

NONLOCAL PIECEWISE DAMAGE MODEL FOR TWO-PHASE MATERIALS

Accumulation of damage leads to strain softening and loss of ellipticity. The local approach, stating that in the absence of thermal effects, stresses in a material at a point are completely determined by the deformation and the deformation history at that point, may result in a physically unacceptable localization of the deformation. To remedy the situation, a number of approaches have been developed to limit strain localization and to circumvent mesh sensitivity associated with strain softening (deVree et al., 1995). One of these approaches is based on the nonlocal damage theory (Bazant, 1991; Bazant and Pijaudier-Cabot, 1988), the essence of which is to smear solution variables causing strain softening over the characteristic volume of the material.

Following the concept proposed by Bazant and Pijaudier-Cabot (1988) the nonlocal damage parameter $\tilde{\omega}(x)$ is defined as (Fish and Yu, 2001b):

$$
\tilde{\omega}(x) = \frac{1}{|\Theta_{ijkl}|} \Theta_{ijkl} \varphi(y) \omega^{(\eta)}(y, x) d\Theta
$$

(31)

where $\varphi(y)$ is a weight function; $\Theta_{ijkl}$ is the characteristic volume in the present manuscript, the representative volume element (RVE) is defined as the maximum required from the statistically homogeneity with which the local periodicity assumption is valid. It is further assumed that the microscopic damage distribution function $\varphi^{(\eta)}(y)$ introduced in (25) is a piecewise function in two-phase (matrix and reinforcement) composites, i.e., it is continuous within the domain of microphase, but vanishes elsewhere, i.e.

$$
\varphi^{(\eta)}(y) = \begin{cases} 
\varphi^{(\eta)}(y) & \text{if } y \in \Theta^{(\eta)} \\
0 & \text{otherwise}
\end{cases}
$$

(32)

where $\eta = m, f$; $m$ and $f$ denote matrix and reinforcement phase respectively; $\varphi^{(\eta)}(y)$ is a $C^\infty$ continuous function in $\Theta^{(\eta)}$, $\varphi^{(\eta)}(x)$ as defined in (25) represents phase average damage variable.

Further define the weight function in (31) as

$$
\varphi(y) = \mu^{(\eta)} \varphi^{(\eta)}(y)
$$

(33)

where the constant $\mu^{(\eta)}$ is determined by the orthogonality condition

$$
\frac{\mu^{(\eta)}}{|\Theta_{ijkl}|} \Theta_{ijkl} \varphi^{(\eta)}(y) \omega^{(\eta)}(y, x) d\Theta = \delta_{\eta, \eta}, \quad \eta = m, f
$$

(34)

and $\delta_{\eta, \eta}$ is Kronecker delta. Substituting (25) and (32)-(34) into (31) provides the motivation for the specific choice of the weight function

$$
\tilde{\omega}(x) = \frac{1}{|\Theta_{ijkl}|} \Theta_{ijkl} \varphi^{(\eta)}(y)^2 \omega^{(\eta)}(x) d\Theta = \omega^{(\eta)}(x)
$$

(35)

which shows that $\omega^{(\eta)}$ has a meaning of the nonlocal phase damage parameter.

The average strains in each subdomain in RVE are obtained by integrating (29) over $\Theta^{(\eta)}$:

$$
e_{ij}^{(\eta)}(x) = \frac{1}{|\Theta^{(\eta)}|} \Theta^{(\eta)} e_{ij} d\Theta
$$

(36)

$$
e_{ij}^{(\eta)} = A_{ijkl}^{(\eta)} \varepsilon_{ijlkl} + G_{ijkl}^{(\eta)} \theta_{ijkl} T + O(\zeta)
$$

(37)

where

$$
A_{ijkl}^{(\eta)} = \frac{1}{|\Theta^{(\eta)}|} \Theta^{(\eta)} A_{ijkl} d\Theta
$$

(38)

The overall strains defined in (18) can be simplified to

$$
\varepsilon_{ij}^{(\lambda)} = \frac{v^{(m)} e_{ij}^{(m)} + v^{(f)} e_{ij}^{(f)}}{\sum_{\eta=m,f} v^{(\eta)} e_{ij}^{(\eta)}}
$$

(39)

where $v^{(\eta)} = |\Theta^{(\eta)}|/|\Theta|$, $\eta = m, f$ is the volume fraction of each micro-phases in RVE. To construct the nonlocal constitutive relation between the phase averages, the local average stress in $\Theta^{(\eta)}$ is defined as:

$$
\sigma_{ij}^{(\eta)} = \frac{1}{|\Theta^{(\eta)}|} \Theta^{(\eta)} \sigma_{ij}^{(\eta)} d\Theta
$$

(40)

By combining (11), (25), (32), (36)-(38), the phase average stress is given as

$$
\sigma_{ij}^{(\eta)} = \left( L_{ijmn}^{(\eta)} - \omega^{(\eta)} G_{klmn}^{(\eta)} L_{ijkl}^{(\eta)} \right) \varepsilon_{ij}^{(\eta)}
$$

(41)

$$
N_{ijmn}^{(\eta)} = (T_{ijmn}^{(\eta)} + G_{ijmn}^{(\eta)} D_{mn}^{(\eta)}) - \omega^{(\eta)} (\varepsilon_{ij}^{(\eta)} + \Theta_{ijkl}^{(\eta)} D_{ijkl}^{(\eta)}) L_{ijkl}^{(\eta)}
$$

(42)

$$
T_{ijkl}^{(\eta)} = \left( \frac{1}{|\Theta^{(\eta)}|} \Theta^{(\eta)} \Theta^{(\eta)} A_{ijkl} d\Theta
$$

(43)

and the overall stresses defined in (30) reduce to

$$
\varepsilon_{ij}^{(\lambda)} = \frac{v^{(m)} e_{ij}^{(m)} + v^{(f)} e_{ij}^{(f)}}{\sum_{\eta=m,f} v^{(\eta)} e_{ij}^{(\eta)}}
$$

(44)
The constitutive equation (41) has a nonlocal character in the sense that it represents the relation between phase averages. The response characteristics between the distinct phases are not smeared as the damage evolution law and thermomechanical properties of phases might be considerably different, in particular when damage occurs in a single phase.

It is interesting to note that the locally isotropic damage defined in (3) behaves anisotropically on the global scale as shown in the macroscopic constitutive equation (41). This anisotropy vanishes, however, when the microscopic damage distribution function $\psi^n(y)$ in (25) is simplified as a piecewise constant function

$$\psi^n(y) = \begin{cases} 1 & \text{if } y \in \Theta^n \\ 0 & \text{otherwise} \end{cases}$$

where $\eta = m, f$, and the corresponding weight function is also piecewise constant with $\mu^n = |\Theta^n|/|\Theta|$. 

**DAMAGE EVOLUTION LAW**

Following the strain-based continuum damage theory, the nonlocal isotropic damage state variable $\omega^n$ is assumed to be a monotonically increasing function of nonlocal phase deformation history parameter $\kappa^n$ (Geers, 1997; Ju 1989; Simo and Ju, 1987), which characterizes the ultimate deformation experienced throughout the loading history. In general, the evolution of matrix damage at time $t$ can be expressed as

$$\omega^n(x, t) = f(\kappa^n(x, t))$$

(46)

The value of the nonlocal phase deformation history parameter $\kappa^n$ is determined by the evolution of a scalar measure: the nonlocal phase damage equivalent strain, denoted by $\vartheta^n$,

$$\kappa^n(x, t) = \max \vartheta^n(x, \tau)[\tau \leq t], \kappa_i^n$$

(47)

where the threshold value for damage initiation in the matrix, $\kappa_i^n$, represents the extreme value of equivalent strains prior to the initiation of damage. Equation (47) can be expressed by the Kuhn-Tucker relations

$$\kappa^n \geq 0, \quad \vartheta^n - \kappa^n \leq 0, \quad \kappa^n(\vartheta^n - \kappa^n) = 0$$

(48)

The damage equivalent strain $\vartheta^n$ is related to the damage energy release rate defined as (Simo and Ju, 1987)

$$\vartheta^n = \left(\frac{1}{2} L_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \right)$$

(49)

Since $L_{ijkl}$ is a positive definite fourth order tensor, along with the definition of damage evolution (46), the following inequality exists

$$\omega^n = \kappa^n \frac{\partial f(\kappa^n(x, t))}{\partial \kappa^n} \geq 0$$

(50)

Combining this inequality with Kuhn-Tucker relations (48), the following two conclusions can be drawn: 1) the damage evolution law $f(\kappa^n(x, t))$ is an increasing function of $\kappa^n \in [\kappa_i^n, \kappa_u^n]$ since $\frac{\partial f(\kappa^n(x, t))}{\partial \kappa^n} > 0$,

where $\kappa_u^n$ is the ultimate equivalent strains at rupture; 2) the damage evolution condition can be expressed as

$$\omega^n > 0$$

(51)

elastic process: $\omega^n = 0$

$$\omega^n < 0 \quad \text{or} \quad \omega^n = 0$$

(52)

In accordance with the above thermodynamic considerations, it is possible to construct an appropriate damage evolution law. An extensive review of a variety of damage evolution laws has been reported in (Geers, 1997). In this paper, an arctangent form of evolution law is adopted to ensure regularity of the tangent stiffness matrices in almost completely damaged state $\Phi(x, \beta, \omega^n, \kappa^n, \kappa_0^n)$

$$= \omega^n - \frac{\pi}{2} \tan(\beta^n) = 0$$

(53)

where $\eta = m, f$ for two-phase composites, $\alpha^n$, $\beta^n$ are material parameters; $\kappa_0^n$ denotes the threshold of the strain history parameter beyond which the damage will develop very quickly. For simplicity, $\kappa_0^n = 0$ is assumed for the threshold value defined in (47) for damage initiation.

**NUMERICAL EXAMPLES**

Based on the multi-scale constitutive theory describe in this paper, a numerical example for the thermo-mechanical induced damaging behavior of a composite patch repair is studied. The configuration of the example is shown in Figure 1. A 100.0mm long through-thickness crack with crack tip radius of 0.2mm is imposed to the steel substrate. The thickness of substrate is 15mm, while the thickness of composite patch is 8mm in the overlap area. The length of patch overlap area along the crack side is 300mm and the length across the crack is chosen as 200mm. The patch tapers beyond overlap area are set at the ratio of 8/50. The composite path is made of 5-harness weave of
carbon fiber bundles surrounded by epoxy matrix. The RVE is modeled as a two-phase composites as shown in Figure 1. The weave bundle is assumed to be linear elastic throughout the analysis, and the damage is assumed to accumulate in epoxy matrix only. In this set of numerical examples, the piecewise constant damage distribution function (45) is assumed.

The material properties used in the analysis can be summarized as follows, where the epoxy matrix is isotropic and the fiber bundles are transversely isotropic.

**Steel Plate Substrate**
- Young’s modulus: 207 GPa
- Poisson’s ratio: 0.3
- Yield strength: 235 MPa
- Ultimate strength: 400 MPa
- Ultimate strain: 0.22
- Coefficient of thermal expansion: $10.3 \times 10^{-6} / \text{C}^0$

**Epoxy Matrix in Composite Patch**
- Volume fraction: 0.548
- Young’s modulus: 3.7 GPa
- Poisson’s ratio: 0.35
- Coefficient of thermal expansion: $54.0 \times 10^{-6} / \text{C}^0$

**Carbon Fiber Bundle in Composite Patch**
- Volume fraction: 0.452
- Axial Young’s modulus: 225 GPa
- Transverse Young’s modulus: 220.7 GPa
- Poisson’s ratio: 0.244
- Axial Shear modulus: 89.0 GPa
- Transverse shear modulus: 88.5 GPa
- Axial coefficient of thermal expansion: $0.2 \times 10^{-6} / \text{C}^0$
- Transverse coefficient of thermal expansion: $15 \times 10^{-6} / \text{C}^0$

Figure 2 depicts the damage evolution law for epoxy matrix with $\alpha^{(m)} = 7.0$, $\beta^{(m)} = 10.5$ and $\kappa^{(m)} = 0.28$, which lead to the progressive failure curves for a bulk composite as shown in Figure 3 to Figure 5. In-plane direction is defined on the cross-weave plane, while the out-plane direction is defined to be perpendicular to the weaves as shown in Figure 1. Thermal stress is not taken into account when computing those loading curves in Figure 3 to Figure 5 for the bulk composites.

Figure 3 presents a stress-strain curve for the bulk composites subjected to the unidirectional in-plane tension along one of the weave directions. It shows a rapid loss of stiffness as the damage in the matrix phase accumulates. In the limit as the matrix material is completely damaged, the in-plane loading capacity is provided only by the fiber bundles, which are assumed elastic in the calculation. A stress-strain curve of out-of-plane tension is shown in Figure 4. It can be seen that when the matrix is totally damaged, it fails to transfer the load into the fiber bundles and consequently, the entire load carrying capacity of the bulk woven composite is lost in the out-of-plane (transverse) direction. The shear stress-strain curves for in-plane and out-of-plane directions are given in Figure 5. Since the shear strength is matrix-dominated, these curves demonstrate the similar trend as out-of-plane tension curve.

Results of composite patch repair example are given in Figure 6 to Figure 9. The composite patch is assumed to be perfectly bonded to the substrate plate. Figure 6 and Figure 7 demonstrate the matrix damage distribution on the bottom surface of composite patch when the substrate plate is subjected to 4-point bending without and with thermal effect respectively. It is shown that the critical area with relative large matrix damage is located right above the crack on substrate. The damage is dominated by the crack opening and shearing. Figure 8 depicts the distribution of local out-of-plane stresses in microscopic RVE, which is in the vicinity of the most critical location with the largest matrix damage, as shown in Figure 7, in composite patch. These local stresses are reconstructed through the scale linking relations (11) and (29) where the global strains and damage are used as input.

The improved stress concentrate at crack tip on substrate plate is clearly demonstrated by Figure 9, where the distribution of in-plane crack opening stress on the top surface connecting to the composite patch is presented. The substrate plate is subjected to the unidirectional tension as shown in the figures. It is shown that, due to the local bending induced by stiffness mismatch between composite patch and substrate, the compressive (crack closure) stress appears on the top surface of substrate plate.
Multi-Scale Finite Element Simulation of Progressive Damage in Composite Structures

**Microscopic (Local) Domain**

![RVE for 5-Harness Woven Composite](image)

![Fiber Bundles](image)

**Macroscopic (Global) Domain**

![Carbon-Epoxy Composite Patch (200mmx300mmx8mm)](image)

![Patch Tapering (8mmx50mm)](image)

![Steel Substrate with 100mm Center Crack](image)

**Figure 1. Global and Local Analysis Domain for Composite Patch**

**Figure 2. Damage Evolution Law for Epoxy Matrix**

**Figure 3. In-Plane Tension Loading Curve for Bulk Composites**

**Figure 4. Out-Of-Plane Tension Loading Curve for Bulk Composites**

**Figure 5. Shear Loading Curves for Bulk Composites**
CONCLUSIONS

A multi-scale progressive damage analysis method for composite structures is presented in this paper. Its efficiency lies on the fact that the complex microscopic domain is not explicitly represented in global structure analysis so that the existing FEA procedures is still applicable. On the other hands, the responses in microscopic field, which can provide a reliable indication of failure initiation and growth, can readily be re-constructed at the vicinity of selected hot-spots in global structures using scale linking relations such as (29). The numerical examples demonstrate the capabilities of the presented methods for the simulation of damage growth due to thermo-mechanical loading. A scalar damage variable is defined in microscopic domain (3). This implies that the damage on microscale is assumed to be isotropic. To account for the directionality of microscopic damage, a damage tensor may be used instead and each tensor components may require its unique damage evolution law (Krajcinovic, 1996). As an alternative, a damage distribution function (41) is used herein to taken into account of the directionality of damage in microscopic phases. The resultant macroscopic (homogenized) constitutive equation (41) does show the directionality of damage on macro-scales.

Note that the definition of damage equivalent strain (49) does not distinguish compression and tension. A possible solution is to introduce a weighting function (Fish and Yu, 2001a) to account for the distinction of contribution to damage growth due to the nature of compression and tension strains.

It is also noted that the assumptions of periodicity and uniformity of macroscopic fields which are embedded in formulation, may yield inaccurate solutions in the microscopic vicinity of boundary layers. The remedies to this phenomenon range from changing the RVE size to carrying out an iterative global-local analysis (Fish and Belsky, 1995).
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REFERENCES


